Workshop 2024/04/13
Introduction
Format of the competition (3 stages, number of questions, format of answer, scoring, winners, prizes, etc…)
Few definitions:
Positive Whole Number. Examples 1, 4, 11
Odd number (either positive or negative): any number that is not divisible by 2. Examples 3,-5,17
Even number (either positive, zero, or negative): any number that is divisible by 2. Examples -10,0,32
Prime number: any whole number greater than one that is only divisible by itself and by 1. Examples 2,3,17

Problem 1:
a. Find all pairs (N, M) where N and M are primes and $5N + 11M = 102$
Suppose that the pair $(N, M), N > 0, M > 0$ satisfies $5N + 11M = 222$.
b. What is the largest possible value of $N + M$?
c. What is the smallest possible value of $N + M$?

Answer 1:
a. Check for M: 2 NG, (5,7)
b and c. (40,2), (29,7),(18,12),(7,17), Largest is 40+2=42, smallest is 7+17=24

Factors of a number: any whole number that divides that number with no remainder (note that 1 and the number itself are also factors). Examples 10 has 4 factors namely 1,2,5,10. 12 has 6 namely 1,2,3,4,6,12

Problem 2:
a) How many factors are there of the following two numbers: 48, 360?
b) What is the sum of all the factors of these numbers: 48, 360?

Answer 2:
Number of factors of $p^k \times q^j$ where p, q are primes is $(1 + p) \times (1 + q)$
$48 = 2 \times 2 \times 2 \times 2 \times 3 = 2^4 \times 3$
Number of factors: $(1 + 4) \times (1 + 1) = 5 \times 2 = 10$
List of factors: $1,2,3,4,6,8,12,16,24,48$
$360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5$
Number of factors: $(1 + 3) \times (1 + 2) \times (1 + 1) = 4 \times 3 \times 2 = 24$
List of factors: $1,2,3,4,5,6,8,9,10,12,15,18,20,24,30,36,40,45,60,72,90,120,180,360$
Sum of factors of $p^k \times q^j$ where p, q are primes is $(1 + p + \cdots + p^k) \times \left(1 + q + \cdots + q^j\right)$
$48 = 2^4 \times 3$
Sum of factors: $(1 + 2 + 2^2 + 2^3 + 2^4) \times (1 + 3) = 31 \times 4 = 124$
$360 = 2^3 \times 3^2 \times 5$
Sum of factors: $(1 + 2 + 2^2 + 2^3) \times (1 + 3 + 3^2) \times (1 + 5) = 15 \times 13 \times 6 = 1170$

Fractions, decimal representation of a number, fractions in lowest terms, conversions between representations.

Problem 3: convert $0.41666\cdots$ to fraction.

Answer 3:

$$x = 0.41666\cdots$$

$$100x = 41.666\cdots = 41\frac{2}{3} = \frac{125}{3}$$

$$x = \frac{125}{300} = \frac{5}{12}$$

Numbers with 5 digits. Examples 17801, 70005
Number whose digit sum is 5. Examples 11111, 5, 221

Problem 4:
Consider a positive number N.
a. Suppose that N is divisible by 9 with no remainder, what is the minimum number of digits of N if its digit sum is greater than 90 but less than 100?
b. Suppose that N is divisible by 5 with no remainder. What is the minimum number of digits of N if its digit sum is 99?
c. Suppose that N is divisible by 4 with no remainder. What is the minimum number of digits of N if its digit sum is 97?
d. Suppose that N is divisible by 3 with no remainder. What is the minimum number of digits of N if its digit sum is 96?
e. Suppose that N is divisible by 2 with no remainder. What is the minimum number of digits of N if its digit sum is 95?

Answer 4:
a. 11. For example $N = 99{,}999{,}999{,}999$
b. If all digits are 9 then the minimum number of digits is 11. NG as last digit must be 0 or 5.
So, number of digits must be at least 12. $99 = 94 + 5$ so, smallest is: $N = 499{,}999{,}999{,}995$
c. $97 = 90 + 7$ and the minimum number of digits is 11. Since it is divided Divide by 4 so last 2 digits must be 96 or 88. So, number of digits is 11, and smallest is: $N = 99{,}999{,}999{,}988$
d. $96 = 90 + 6$ and the minimum number of digits is 11. Divide by 3 so largest last digit must be 9. So first digit must be 6. So, number of digits is 11, and smallest is: $N = 69{,}999{,}999{,}999$
e. $95 = 87 + 8$ so last digit is 8, and the minimum number of digits is 11.First digits must be 6.
So, number of digits is 11, and smallest is: $N = 69{,}999{,}999{,}998$

Numbers that define some measurements.

Distance: mm, cm, m, km, astronomical and microscopic distences.

Area and volume: $mm^2, mm^3, cm^2, cm^3, m^2, m^3, km^2, km^3$

Weight or mass : g, kg, ton, larger and smaller masses

Density: for example $\frac{kg}{m^3}$

Time units: $sec, min, h,$ day month, year, larger and smaller time units.

Speed: distance/time: for example $\frac{km}{h}$

Problem 5:

Suppose that light travels at speed of $300{,}000\,\frac{km}{sec}$.

a. How many kilometres will it travel in one minute.

b. If the Earth is $150{,}000{,}000\ km$ away from the sun, how many minutes it takes for the sun light to reach Earth (round the answer to the nearest minute).

c. If Pluto is 6 billion km away from the sun, how many hours it takes the light to travel from the Sun to Pluto.

d. Suppose that one of the nearest stars is located $60{,}000{,}000{,}000{,}000\ km$ away. Approximately, how many years it takes for the Sun light to reach that star?

Answer 5:

a. $300{,}000 \times 60 = 18{,}000{,}000\ km$

b. $\frac{150}{18} = 8.333\ km$

c. In one hour light travels $18{,}000{,}000 \times 60 = 1{,}080{,}000{,}000\ km$

$$\frac{6}{1.08} = 5.555\ hours$$

d. In one year light travels $1{,}080{,}000{,}000 \times 24 \times 360 = 9.3312 \times 10^{12}\ km$

$\frac{60}{9.3312} = 6.43$ years.

Sequences: arithmetic: $\{N, N+K, N+2K, \cdots\}$, geometric: $\{a, ab, ab^2, \cdots\}$.

Examples: $\{N, N+K, N+2K, \cdots\}$, $\{8,6,\frac{9}{2},\cdots\}$.

Problem 6:

a. What is the value of the 10-th term of the following arithmetic sequence: $\{-7,-3,1,\cdots\}$?

a. What is the value of the 6-th term of the following geometric sequence: $\{-1,2,-4,\cdots\}$?

Answer 6:

a. $-7 + 9 \times 4 = 29$

b. $-1 \times 2^5 = -32$

Probabilty of an event. Examples: tossing 3 coins, number of Heads; taking beads out of a bucket, number of beads of certain colour.
Conditional probability. Example tossing 3 coins and we know that a head was tossed at least once.

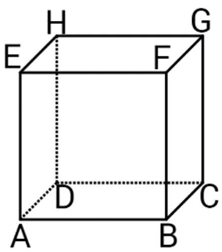
Problem 7:
Suppose that 5 dice are rolled.
a. How many different sums can be rolled?
b. How many of these sums are odd?
c. How many of these sums are even?
d. How many of these sums are multiples of 5?
e. How many of these sums are prime numbers?
f. What is the probability that the sum is 7?
Suppose that 2 fair dice are rolled.
g. if you know that at least one of the dice shows an odd number, what is the probability that the sum is odd?
h. if you know that at least one of the dice shows an odd number, what is the probability that the sum is even?
i. if you know that at least one of the dice shows an even number, what is the probability that the sum is odd?
j. if you know that at least one of the dice shows an even number, what is the probability that the sum is even?

Answer 7:
a. Minimum sum is 5. Maximum sum is 30. Number of sums is 26.
b. $\{5,7,9,\cdots,29\}$, number of sums 13.
c. $\{6,8,10,\cdots,30\}$, number of sums 13.
d. $\{5,10,15,\cdots,30\}$, number of sums 6.
e. $\{5,7,11,\cdots,29\}$, number of sums 8.
f. $\{1,1,1,1,3\}$: choose 1 out of 5. 5 options. $\{1,1,1,2,2\}$: choose 2 out 5. 5X2=10 options. Total 15.
Probabilty: 15/7776
g. $\{x,y\}$, sum odd: $\{1,y\}$ 6 options. $\{2,y\}$ 3 options. $\{3,y\}$ 6 options. and so on. So total number of cases where at least one is odd: $6\times 3+3\times 3=27$. Sum of dice is odd if one is odd and one is even. Total is $3\times 3+3\times 3=18$. So probability is $\frac{2}{3}$.
h. Sum is odd if both are odd. Using the result of g., the probability $\frac{1}{3}$.
i. Same answer as for g: $\frac{2}{3}$.
j. Same answer as for h: $\frac{1}{3}$.

Below is a box with sides x, y, z.
A, C, E, G are the corners of the box.
The lines AB, CG, DH are edges.
$ABFE$ is one of the faces of the box.

Problem 8:

Suppose that the values of the edges of the box below are $x = 2$, $y = 3$, and $z = 4$.
Find the following:
a. What is the volume of the box?
b. How many faces does the box have?
c. How many edges does the box have?
d. How many corners does the box have?
e. What are the areas of each of its face? And, what is the total area of all its faces?
f. What is the sum of all its edges?



Answer 8:

a. $2 \times 3 \times 4 = 24$.
b. 6.
c. 12.
d. 8.
e. $2 \times 3 = 6,\ 2 \times 4 = 8, 3 \times 4 = 12.$ So total area is: $(6 + 8 + 12) \times 2 = 52$.
f. $(2 + 3 + 4) \times 2 = 18$.

Problem 9:

On the ground there is a big container of water that is initially empty. The container can be filled with 3 taps, (that can operate at the same time if desired), named A, B, and C.
Tap A operating alone can fill it up in 6 hours, tap B operating alone can fill it up in 8 hours, and tap C operating alone can fill it up in 24 hours. How many hours will it take to fill the container up if all 3 taps are turned on?

Answer 9:

$$\frac{1}{6} + \frac{1}{8} + \frac{1}{24} = \frac{1}{x}$$

Simplify by multiplication by $24x$.

$$4x + 3x + x = 24.$$

$$x = \frac{24}{8} = 3.$$

Problem 10:

a. What is the sum of all angles (in degrees) of a regular polygon with 11 sides?
b. What is the size (in degrees) of each of the angles of a regular polygon with 12 sides?
c. What is the largest number of sides of a regular polygon if all of its angles are whole numbers (in degrees)?
d. If the value, (in degrees), of each of the angles of the regular polygon is a prime number, what is minimum number of sides that such a polygon can have?

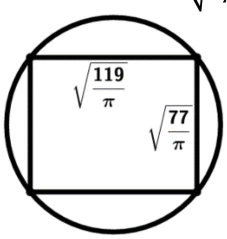
Answer 10:
a. By connecting the corners of the triangle, we can generate 9 triangles.
From below, sum of all angles is 1620 deg.
$$\frac{9\times180}{11} = \frac{1620}{11} = 147.27\cdots$$
b. $\frac{10\times180}{12} = \frac{1800}{12} = 150$
c. The largest angle is 179 degrees. The supplementary angle to it is 1 degree so the number of sides is 360.
d. Supplementary angles to angles of the polygon need to be a whole number that divide 360:
$$\{1,2,3,4,5,6,8,9,10,12,15,18,20,24,30,36,40,45,60,72,90,120\}$$
Angles of the polygon: 179,178,177,176,175,172,171,170,168,165,162,160,156,150,144,140,135,120,108,90,60
179 is the only prime number.

Powers, expressions of N^k. Examples

Problem 11:
How many positive numbers N satisfy $N^4 < 1{,}000{,}000$?

Answer 11:
$$N^2 < 1{,}000$$
N<32

Problem 12:
A rectangle with sides $\sqrt{\frac{119}{\pi}}$ and $\sqrt{\frac{77}{\pi}}$ is inscribed in a circle. What is the area of the circle?



Answer 12:
Use the Pythagorean theorem for a right triangle $a^2 + b^2 = c^2$.
The diameter of the circle, d, satisfies:
$$d^2 = \frac{119}{\pi} + \frac{77}{\pi} = \frac{196}{\pi}.$$
The radius of the circle, r, is $\frac{d}{2}$.
Thus, the area of the circle is:
$$\pi r^2 = \frac{\pi d^2}{4} = \frac{\pi 49}{\pi} = 49.$$

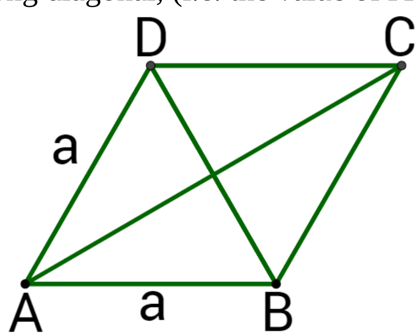
Problem 13:
For every 1400 boys members of the Canadian Arts and Science Club there are 2023 girls that are members of the club. What percentage of the club membership are the girls? Round your answer to the nearest whole number.

Answer 13:
Percentage of girls is given by:
$$100 \times \frac{2023}{2023+1400} = 100X\frac{7\times 289}{7\times(289+200)} = 100X0.591\cdots = 59.1\cdots.$$
So, the percentage of girls is 59.

Problem 14:
Two teams, A and B, compete in a basketball championship. The probability of Team A to win a game is 80%, and the probability of Team B to win a game is 20%, (no ties). The first team to win 3 games in total wins the championship. What is the probability that it will take only 3 games to decide the championship? Express the answer as a fraction in lowest terms.

Answer 14:
Either team A wins 3 times in a row, or team B wins 3 times in a raw.
So, the probability is:
$$(\frac{20}{100})^3 + (\frac{80}{100})^3 = (\frac{1}{5})^3 + (\frac{4}{5})^3 = \frac{1}{125} + \frac{64}{125} = \frac{65}{125} = \frac{13}{25}.$$

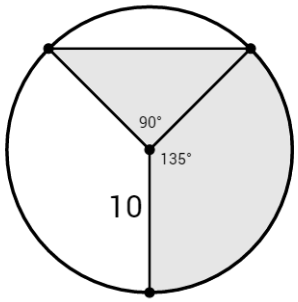
Problem 15:
Below is a rhombus with sides $a = 7$. Its short diagonal satisfies $BD = a = 7$. What is the square value of the long diagonal, (i.e. the value of AC^2) ?



Answer 15:
In a rhombus BD is perpendicular to AC. Also, the 2 diagonals are bisected.
Using the Pythagorean Theorem $(\frac{AC}{2})^2 + (\frac{a}{2})^2 = a^2$.
So, $(\frac{AC}{2})^2 = a^2 - \left(\frac{a}{2}\right)^2 = \frac{3}{4}a^2 = \frac{3}{4}\times 49 = \frac{147}{4}$.
So, $AC^2 = 4 \times \frac{147}{4} = 147$.

Problem 16:

The shaded section of the circle of radius 10 consists of a right triangle and a sector of $135°$. Find the area of the shaded section. Use $\pi = 3.14$, and round your answer to the nearest whole number.



Answer 16:

The shaded area is consisted of a right triangle and a sector of the circle.

The area of the right triangle is:

$$\frac{10\times10}{2} = 50.$$

The area of the $135°$ sector is (using the approximation):

$$3.14 \times 10^2 \times \frac{135}{360} = 314 \times \frac{3}{8} = 117.75.$$

So, the area of shaded section is:

$$50 + 117.75 = 167.75.$$

Rounding the area to the nearest whole number the requested solution is: 168.

Problem 17:

$\frac{1}{x} + \frac{1}{2x} + \frac{1}{3x} = 3$. Express x as a fraction in lowest terms.

Answer 17:

$6x$ is a common denominator so $\frac{6+3+2}{6x} = 3$.

So, $11 = 6x \times 3 = 18x$, so $x = \frac{11}{18}$.

Problem 18:

Jill drives a fuel-efficient car that consumes, on average, 6 litres of fuel per hour. When she started driving, the fuel tank was full. After driving T hours she stopped and added 10 litres of fuel so that the tank was 85% full. Then, she drove $\frac{T}{2}$ hours, stopped again, and filled the tank with 17 more litres of fuel so that the fuel tank was full again. How many liters of fuel can a full tank hold? Round your answer to the nearest whole number.

Answer 18:

In T hours fuel consumption was $6T$ litres. In $\frac{T}{2}$ hours the fuel consumption was $3T$ litres.

Thus, total fuel during the entire drive was $6T + 3T = 9T$ litres.

Total fuel added was $10 + 17 = 27$ litres.

So, $9T = 27$, and thus $T = 3$ hours.

Define V to be the volume of the fuel tank (in litres).

Thus, $0.85V = V - 6T + 10 = V - 18 + 10 = V - 8$.

Thus, $0.15V = 8$, so $V = \frac{8}{0.15} = \frac{800}{15} = \frac{160}{3} = 53.333\cdots$ litres.

Rounding to the nearest whole number: $V = 53$.

Problem 19:

The parliament proposes an increase of 400% to the current carbon tax to a new tax rate of $120 per tonne. What is the current tax rate per tonne (in $)?

Answer 19:

Increasing by 400% is the same as multiplying by 5.

Thus, the current tax rate is: $\frac{120}{5}$=24.

Problem 20:

In how many ways can you pay 80 cents using any combination of 5, 10, and 25 cent coins?

Answer 20:

Divide the possibilities of the sum of 80 cents into the following groups.

a) 3 0.25$, b) 2 0.25$, c) 1 0.25$, or d) 0 0.25$.

Check each of the groups for the number of options of 0.10$.

a) 0.75$ in 0.25$ coins: only 1 option for 0.10$: 0 coins of 0.10$.

b) 0.50$ in 0.25$ coins: 4 options for 0.10$: $0,1,2,$ or 3.

c) 0.25$ in 0.25$ coins: 6 options: $0,1,2,3,4,$ or 5.

d) 0.00$ in 0.25$ coins: 9 options: $0,1,2,3,4,5,6,7,$ or 8.

Total number of ways is:

$1 + 4 + 6 + 9 = 20$.

Problem 21:

N is the smallest positive whole number such that all the following conditions are satisfied: $\{a, b, c, d, e, f\}$ is a set of 6 different primes, $N = a + b + c = d + e + f$, $a < b < c$, and $d < e < f$. What is the maximum possible value of $c - a$?

Answer 21:

Note that N must be odd based on the condition of the value of N and that the sum of the 6 different numbers of the set is $2N$. Also, since N is odd then the only even prime number, 2, is not a part of the set.
Set of the smallest, odd primes is the set:

$\{3,5,7,11,13,17\}$.

But, $3+5+7+11+13+17=56=2\times 28$. So, this set does not satisfy the condition as N must be odd.
Next set of odd primes with smallest sum is the set:

$\{3,5,7,11,13,19\}$.

Thus, $3+5+7+11+13+19=58=2\times 29$.
The only way to divide this set into 2 groups of 3 members each with equal sum of all members of each group is:

$\{3,7,19\},\{5,11,13\}$, as $29=3+7+19=5+11+13$.

Thus, maximum possible value of $c-a$ is:

$c-a=19-3=16$.

Problem 22:
Dan read a 650 page book in the following way. On the first day he read every second page of the book starting at page 1 (i.e. he read pages 1, 3, 5, and so on). On the second day he read every third page of the book starting at page 1 (i.e. he read pages 1, 4, 7, and so on). How many of the pages did he read twice?

Answer 22:

Dan read twice every 6-th page starting at page 1. So, he read twice pages $1,7,13,\cdots,649$. So he read twice: $\frac{654}{6}=109$ pages.

Problem 23:

What is the sum of all factors of 2023? (**Hint**: 2023 is not a prime number).

Answer 23:

2100 is divisible by 7, so 2030 is divisible 7, so 2023 is divisible by 7.

Thus, $\frac{2023}{7}=289=17^2$, so, $2023=17^2\times 7$.

Using the formula for the sum of all factors we get:

$(1+17^1+17^2)\times(1+7^1)=(1+17+289)\times(1+7)=307\times 8=2456$.

Problem 24

A triangle has sides L, M, and N, where $0<L<M<N<12$ are all whole numbers. The perimeter of the triangle is P. How many different values of P are there?

Answer 24:

For any triangle with sides X, Y, and Z: $X + Y > Z$:

So, the triangle with whole numbers that satisfy the requirements, $(L = 1, M, N)$ is not possible.

So the minimum possible value of P is $P = 9$ of a triangle (L=2,M=3,N=4).

$P = 10$ is not possible: $(L = 2, M = 3, N = 5)$ is not a triangle because $2 + 3 = 5$. Also not acceptable are: $(L = 2, M = 4, N = 4)$, and $(L = 3, M = 3, N = 4)$, ($L = M$ or $M = N$ is not acceptable).

$P = 11$ ok, $(2,4,5)$.

$P = 12$ ok, $(3.4.5)$.

$P = 13$ ok, $(3,4,6)$.

$P = 14$ ok, $(3,5,6)$.

$P = 15$ ok, $(3,5,7)$ and more.

And, so on, …

Max perimeter is $P = 30$, $(9,10,11)$.

Summary of P values:

$\{9,11,12,13,\cdots,30\}$. So total is 21 different values of P.

Problem 25:

$N + (N + 1) + (N + 2) + \cdots + 2023 = 94 \times 1000$, and N>0. What is the Value of N?

Answer 25:

Note that $N > 0$. An easy way to solve is simply by observing that 94000 is sum of 47 numbers around the value of 2000 each. Thus, a simple way to solve will be to simply take such 47 consecutive numbers and check. If a student does not see this right away, then some sweating is required.

Use the summation formula of consecutive integers:

$N + (N + 1) + (N + 2) + \cdots + 2023 = (N + 2023)(2024 - N)/2$.

So, from the equation above, $\frac{(N+2023)(2024-N)}{2} = 94 \times 1000 = 47 \times 2000$.

Thus, $(N + 2023)(2024 - N) = 47 \times 4000.$

Note: in secondary school students learn that this equation, (quadratic equation), has either 0, 1, or 2 solutions for the unknown N. We do not expect students in elementary school to know how to solve this equation using the quadratic formula. But, from the way the question is phrased, the sum is of is of consecutive integers, and thus N is either a positive whole number or a negative integer. Thus, we need to assume that there is a positive whole number solution to the above equation. 47 is a prime number, so, for N positive, it must be a factor of exactly one term of the multiplication above, (assuming that both terms are positive).

It is now the time to make an educated guess based on the information above.

So, try and plug $2024 - N = 47$.

Thus, $N = 1977$.

It is immediately clear, based on this guess, that $N + 2023 = 4000$, so, $N = 1977$ is a solution. As mentioned above.

As mentioned above, the quadratic equation, indeed, has another solution. But the other solution is negative: $N = -1976$.

Problem 26:

Frank wrote down the sum of the digits of every number from 1 to 1000. (Examples: for the number 2 he wrote 2, for the number 24 he wrote 6 because $2+4=6$, for the number 200 he wrote 2 because $2+0+0=2$, and for the number 550 he wrote 10 because $5+5+0=10$). How many times did he write the digit 0?

Answer 26:

Consider all possible cases where the digit sum of a positive number contains the digit 0 .

The number with the largest possible digit sum is: 999, because $9+9+9=27$.

Thus, the only possible candidate sums that contain the digit 0 are the digit sums:

A) 10, and B) 20 .

Case A: Sum is 10

For 3-digit numbers,

No xxx and no x00 where the digit x is not 0.

For xx0 (or x0x), $x+x+0=10$, we have the numbers 550 and 505, so **2** times the digit 0 is written.

For xyo, yx0, x0y, y0x, x and y are different digits and none is 0, we have the following possible sums (4 times for each selection):

$9+1+0, 8+2+0, 7+3+0, 6+4+0$. So total number that the digit 0 is written is **16**.

For xyy, 3 times for each of: $8+1+1, 6+2+2, 4+3+3, 2+4+4$. So total is **12.**

For xyz, 6 times for each of: $7+2+1, 6+3+1, 5+4+1, 5+3+2$. So total is **24.**

For 2-digit numbers,

No x0

For xx, 1 time for $5+5$. Total is **1.**

For xy, 2 times for each of: $9+1, 8+2, 7+3, 6+4$. Total is **8.**

Total for digit sum of 10: $2+16+12+24+1+8=63$.

Case B: sum is 20

No xxx

No xx0

No x00

No xy0

For xyy, 3 times for each of $8+6+6, 6+7+7, 4+8+8, 2+9+9$. Total is **12.**

For xyz, 6 times for each of $9+8+3, 9+7+4, 9+6+5, 8+7+5$. Total is **24.**

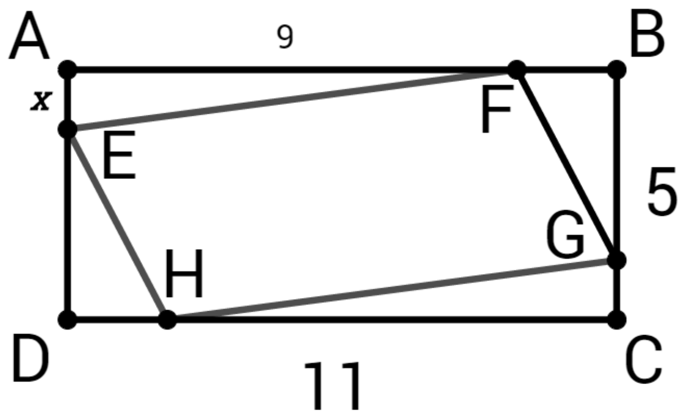
No x0, xx, xy, x

Total for digit sum of 20: $12+24=36$

Thus, $63+36=99$

Problem 27:

$ABCD$ is a rectangle with sides 11 and 5. $EFGH$ is a parallelogram. $AE = x, AF = 9$. The area of $EFGH$ is $\frac{2}{3}$ of the area of $ABCD$. What is the area of ΔDEH? Express the answer as a fraction in lowest terms.



Answer 27:

$EFGH$ is a parallelogram. Note that EF is not necessarily perpendicular to EH.

Area of $ABCD$ is 55 so the sum of the areas of AEF, EDH, HCG, and GBF is $\frac{55}{3}$.

$FB = AB - AF = 11 - 9 = 2$.

Thus, $9x + 2(5 - x) = \frac{55}{3}$.

$9x + 10 - 2x = \frac{55}{3}$

Multiply by 3: $21x + 30 = 55$, so 21x=55-30=25.

Thus, $x = \frac{25}{21}$.

Area of triangle DEH is $\frac{2(5-x)}{2} = 5 - x = 5 - \frac{25}{21} = \frac{105-25}{21} = \frac{80}{21}$

Problem 28:

There is a pile of 7 cards numbered $1,2,3, \cdots ,7$ on the table. Gloria takes 3 different cards at random from the pile and writes down the sum of these 3 cards. What is the probability that the sum is a multiple of 3? Express the answer as a fraction in lowest terms.

Answer 28:

As shown in many workshop before, there is a formula for the number of ways to select 3 items out of 7 items:
$\frac{7!}{3!4!} = \frac{5\times6\times7}{2\times3} = 35$.

Systematically it can easily be shown that there are 13 possible ways for the sum to be a multiple of 3:
$\{123,126,135,147,156,234,237,246,267,345,357,456,567\}$.

So, the probability is $\frac{13}{35}$.

Problem 29:
Eric takes 4 times longer to paint a ceiling than to paint a wall. He charges 20% more per hour to paint a ceiling than to paint a wall. Eric painted 5 ceilings and 12 walls. His hourly charge per wall was $40 and his total earning was $2520. How many hours did he work in total?

Answer 29:

Define T to be the time in hours to paint a wall.

Thus, $4T$ is the time in hours to paint a ceiling.

$40T$ is the earning in $ to paint a wall.

Thus, $4 \times 1.2 \times 40T = 192T$ is the earning in $ to paint a ceiling.

Thus, for 12 walls and 5 ceilings the earning (in $) is:

$$2520 = 12 \times 40T + 5 \times 192T = (480 + 960)T = 1440T.$$

Thus, $T = \frac{2520}{1440} = 1.75$ is the time in hours to paint a wall.

Total time (in hours) is then: $1.75 \times 12 + 1.75 \times 5 \times 4 = 21 + 35 = 56$.

Check the solution: $21 \times 40 + 35 \times 48 = 840 + 1680 = 2520$.